DAWSON COLLEGE
MATHEMATICS DEPARTMENT
Calculus I (Science)

201-NYA-05
Winter 2022
Final Examination
May 24th, 2022
Time Limit: 3 hours

Name: ______________________

ID#: ______________________

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- ▷ This exam contains 16 pages (including this cover page) and 15 problems. Check to see if any pages are missing.
- ▷ Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- ▷ Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- ▷ You are only permitted to use the **Sharp EL-531XT**, **Sharp EL-531XG** or **Sharp EL-531X** calculator.
- ▷ This examination booklet must be returned intact.
- ▷ Good luck!

Question	Points	Score
1	12	
2	4	
3	6	
4	6	
5	6	
6	5	
7	12	
8	4	
9	4	
10	16	
11	5	
12	4	
13	5	
14	6	
15	5	
Total:	100	

[12 pts] 1. Evaluate the following limits (do not use L'Hospital's Rule):

(a) $\lim_{x \to -2} \dfrac{\dfrac{1}{x+3} - 1}{x+2}$ $= \lim_{x \to -2} \dfrac{\dfrac{1-(x+3)}{x+3}}{x+2} = \lim_{x \to -2} \dfrac{1-x-3}{(x+2)(x+3)} = \lim_{x \to -2} \dfrac{-\cancel{(x+2)}}{\cancel{(x+2)}(x+3)}$

$= \dfrac{-1}{-2+3} = \boxed{-1}$

(b) $\lim_{\theta \to 1} \dfrac{\sin(\pi(\theta - 1))}{\theta^2 - 1}$ $= \lim_{\theta \to 1} \dfrac{\sin(\pi(\theta-1))}{(\theta-1)(\theta+1)} = \lim_{\theta \to 1} \dfrac{1}{\theta+1} \cdot \lim_{\theta \to 1} \dfrac{\sin(\pi(\theta-1))}{(\theta-1)} \cdot \dfrac{\pi}{\pi}$

$= \dfrac{1}{2} \cdot \pi \lim_{\theta \to 1} \dfrac{\sin(\pi(\theta-1))}{\pi(\theta-1)} = \dfrac{\pi}{2} \cdot 1 = \boxed{\dfrac{\pi}{2}}$

(c) $\lim_{\theta \to \infty} \dfrac{\cos(\pi\theta)}{\theta^2}$

$-1 \leq \cos(\pi\theta) \leq 1$

$-\dfrac{1}{\theta^2} \leq \dfrac{\cos(\pi\theta)}{\theta^2} \leq \dfrac{1}{\theta^2}$

$\theta \to \infty$: $-\dfrac{1}{\theta^2} \to 0$ and $\dfrac{1}{\theta^2} \to 0$

By the Squeeze Theorem, $\lim_{\theta \to \infty} \dfrac{\cos \pi\theta}{\theta^2} = \boxed{0}$

(d) $\displaystyle\lim_{x\to -1^-}\left[\frac{2}{|x+1|}\left(\frac{1}{3}-\frac{1}{x+4}\right)\right] = \lim_{x\to -1^-}\left(\frac{2}{-(x+1)}\left(\frac{1}{3}-\frac{1}{x+4}\right)\right)$

$|x+1| = -(x-1)$ for $x<-1$

$$= \lim_{x\to -1^-}\left[\frac{-2}{(x+1)}\left(\frac{x+4-3}{3(x+4)}\right)\right] = \lim_{x\to -1^-}\frac{-2\,\cancel{(x+1)}}{3\cancel{(x+1)}(x+4)}$$

$$= \frac{-2}{3(-1+4)} = \boxed{\frac{-2}{9}}$$

[4 pts] 2. Find te following limit using L'Hospital's Rule:

$\displaystyle\lim_{x\to 0}\underbrace{\left(e^{2x}+x\right)^{\frac{1}{x}}}_{y}$

$$\ln y = \frac{1}{x}\ln(e^{2x}+x)$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(e^{2x}+x)}{x} \overset{HR}{=} \lim_{x\to 0}\frac{\frac{2e^{2x}+1}{e^{2x}+x}}{1} = \lim_{x\to 0}\frac{2\cdot e^{2x}+1}{e^{2x}+x}$$

$$= \frac{2\cdot 1+1}{1+0} = 3$$

$$\lim_{x\to 0}\left(e^{2x}+x\right)^{1/x} = \boxed{e^3}$$

[6 pts] 3. Consider the function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x+2} & \text{if } x < -1 \\ 3x & \text{if } -1 \leq x \leq 2 \\ x^2 & \text{if } 2 < x \end{cases}$$

For which value(s) of x is $f(x)$ not continuous?
Justify your answer by referring to the definition of continuity.

f is not defined at $x = -2 \Rightarrow$ not continuous at $x = -2$

f is continuous on $(-\infty,-2), (-2,-1), [-1,2], (2,\infty)$ (defined rational or polynomial function)

at $x = -1$: $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2-4}{x+2} = \frac{(-1)^2-4}{-1+2} = -3$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 3x = -3$

$\lim_{x\to -1} f(x) = -3 = f(-1)$

$\Rightarrow$ continuous at -1

at $x = 2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3x = 3\cdot 2 = 6$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 2^2 = 4$

$\neq$ f is not continuous at 2

$\Rightarrow$ $\boxed{f \text{ is discontinuous at } x = -2 \text{ and } x = 2}$

[4 pts] 4. (a) Using only the limit definition of the derivative, find $f'(x)$ for $f(x) = \sqrt{4+x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{4+x+h} - \sqrt{4+x}}{h} \cdot \frac{\sqrt{4+x+h} + \sqrt{4+x}}{\sqrt{4+x+h} + \sqrt{4+x}}$$

$$= \lim_{h \to 0} \frac{(4+x+h) - (4+x)}{h(\sqrt{4+x+h} + \sqrt{4+x})} = \lim_{h \to 0} \frac{\cancel{4} + \cancel{x} + h - \cancel{4} - \cancel{x}}{h(\sqrt{4+x+h} + \sqrt{4+x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+x+h} + \sqrt{4+x})} = \frac{1}{\sqrt{4+x+0} + \sqrt{4+x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{4+x}}}$$

[2 pts] (b) Use the rules of differentiation to check your answer.

$$f(x) = (4+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(4+x)^{-1/2}(0+1)$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{4+x}}}$$

5. Given the equation $y^3 + 3xy = 1 - x^3$,

[4 pts] (a) Find $\dfrac{dy}{dx}$

$$\frac{d}{dx}(y^3 + 3xy) = \frac{d}{dx}(1 - x^3)$$

$$3y^2 \cdot y' + 3y + 3xy' = 0 - 3x^2$$

$$\cancel{3}\left[(y^2 + x)\, y'\right] = \cancel{3}(-x^2 - y)$$

$$\boxed{\frac{dy}{dx} = -\frac{x^2 + y}{y^2 + x}}$$

[2 pts] (b) Find an equation of the tangent line to the graph at the point $(2, -1)$.

Take $x = 2$ and $y = -1$

$\mapsto$ Slope $= m = -\dfrac{2^2 - 1}{(-1)^2 + 2} = -\dfrac{3}{3} = -1$

$$y - (-1) = -1(x - 2)$$

$$y = -x + 2 - 1$$

$$\boxed{y = -x + 1}$$

[5 pts] 6. Consider the function $f(x) = (x-1)^2(x+1)^{\frac{2}{3}}$
Find the x-values where f has horizontal tangent line(s)

$$f'(x) = 2(x-1)(x+1)^{2/3} + (x-1)^2 \cdot \frac{2}{3}(x+1)^{-1/3}$$

$$= \frac{2(x-1)}{3(x+1)^{1/3}}\left(3(x+1)+(x-1)\right)$$

$$= \frac{2(x-1)(3x+3+x-1)}{3(x+1)^{1/3}}$$

$$= \frac{2(x-1)(4x+2)}{3(x+1)^{1/3}} = \frac{4(x-1)(2x+1)}{3(x+1)^{1/3}}$$

$$f'(x) = 0 \iff x-1=0 \text{ or } 2x+1=0$$

$$\iff \boxed{x = 1 \text{ or } x = -\frac{1}{2}}$$

[12 pts] 7. Find the derivative of each of the following functions. Do not simplify the answer.

(a) $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$$f'(x) = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x - (-e^{-x}))}{(e^x - e^{-x})^2}$$

(b) $g(x) = x^3 \tan x + \arctan(x^3)$

$$g'(x) = 3x^2 \tan x + x^3 \sec^2 x + \frac{3x^2}{1 + (x^3)^2}$$

(c) $y = 7\ln(x - x^2 + 1) - \log_7(3 - x)$

$$\frac{dy}{dx} = \frac{7(1-2x)}{x - x^2 + 1} - \frac{-1}{(3-x)\ln 7}$$

[4 pts] 8. Use logarithmic differentiation to find the derivative of the function $h(x) = (\cos x)^x$

$$\ln(h(x)) = x \ln(\cos x)$$

$$\frac{h'(x)}{h(x)} = 1 \ln(\cos x) + x \cdot \frac{-\sin x}{\cos x}$$

$$\boxed{h'(x) = \left(\ln(\cos x) - x \tan x\right)(\cos x)^x}$$

[4 pts] 9. Show that $y = x\cos x + \sin x$ satisfies the differential equation $xy''' + y'' + xy' + 3y = 0$

$$y' = 1\cos x - x\sin x + \cos x = 2\cos x - x\sin x$$

$$y'' = -2\sin x - \sin x - x\cos x = -3\sin x - x\cos x$$

$$y''' = -3\cos x - \cos x - x(-\sin x) = -4\cos x + x\sin x$$

$$xy''' + y'' + xy' + 3y = x(-4\cos x + x\sin x) - 3\sin x - x\cos x + x(2\cos x - x\sin x) + 3(x\cos x + \sin x)$$

$$= -4x\cos x + \cancel{x^2\sin x} - \cancel{3\sin x} - x\cos x + 2x\cos x - \cancel{x^2\sin x} + 3x\cos x + \cancel{3\sin x}$$

$$= -3x\cos x + 3x\cos x = 0 \quad \checkmark$$

[16 pts] 10. Consider the function and its derivatives

$$f(x) = \frac{x^3 - 8}{x^3 + 8} \qquad f'(x) = \frac{48x^2}{(x^3 + 8)^2} \qquad f''(x) = \frac{-192x(x^3 - 4)}{(x^3 + 8)^3}$$

(a) Find the domain, intercepts and asymptotes of $f(x)$ (if they exist).

(b) Find the intervals where $f(x)$ is increasing/decreasing and the points where local maxima and minima occur (if they exist).

(c) Find the intervals where $f(x)$ is concave upward/downward and the points of inflection (if they exist).

(d) Sketch the graph of $f(x)$
Show clearly all important points.

(a) * $x^3+8=0 \iff x^3=-8 \iff x=-2 \Rightarrow$ Domain: $\mathbb{R}-\{-2\}$

* $f(0)=\frac{0-8}{0+8}=-1 \rightarrow$ y-intercept: $(0,-1)$
* $f(x)=0 \iff x^3=8 \iff x=2 \Rightarrow$ x-intercept: $(2,0)$
* $\lim\limits_{x\to\pm\infty} \frac{x^3-8}{x^3+8} = \lim\limits_{x\to\pm\infty} \frac{1-8/x^3}{1+8/x^3} = 1 \Rightarrow$ H.A. $y=1$
* $\lim\limits_{x\to -2^-} f(x) = \frac{-16}{0^-} = \infty \qquad \lim\limits_{x\to -2^+} f(x) = \frac{-16}{0^+} = -\infty \Rightarrow$ V.A. $x=-2$

(b) $f'(x)=0 \iff x^2=0 \iff x=0 \qquad f'(x)$ DNE $\iff x^3+8=0 \iff x=-2$

$f'(x) \geq 0$ on $\mathbb{R} \Rightarrow f$ is increasing on $(-\infty,-2)$ and $(-2,\infty)$

No local extrema

(c) $f''(x)=0 \iff x(x^3-4)=0 \iff x=0$ or $x=\sqrt[3]{4}$ $\qquad f(\sqrt[3]{4}) = \frac{4-8}{4+8} = \frac{-4}{12} = \frac{-1}{3}$

$f''(x)$ DNE $\iff x^3+8=0 \iff x=-2$

x	$-\infty$		-2		0		$\sqrt[3]{4}$		2		∞
$f'(x)$		$+$	‖		0		$+$				
$f''(x)$		$+\ +$ ∪	‖	$-\ -$ ∩		$+\ +$ ∪			$-\ -$ ∩		
$f(x)$	1	↗ ∞	‖	$-\infty$ ↗	-1	↗	$-1/3$	↗	0	→	1

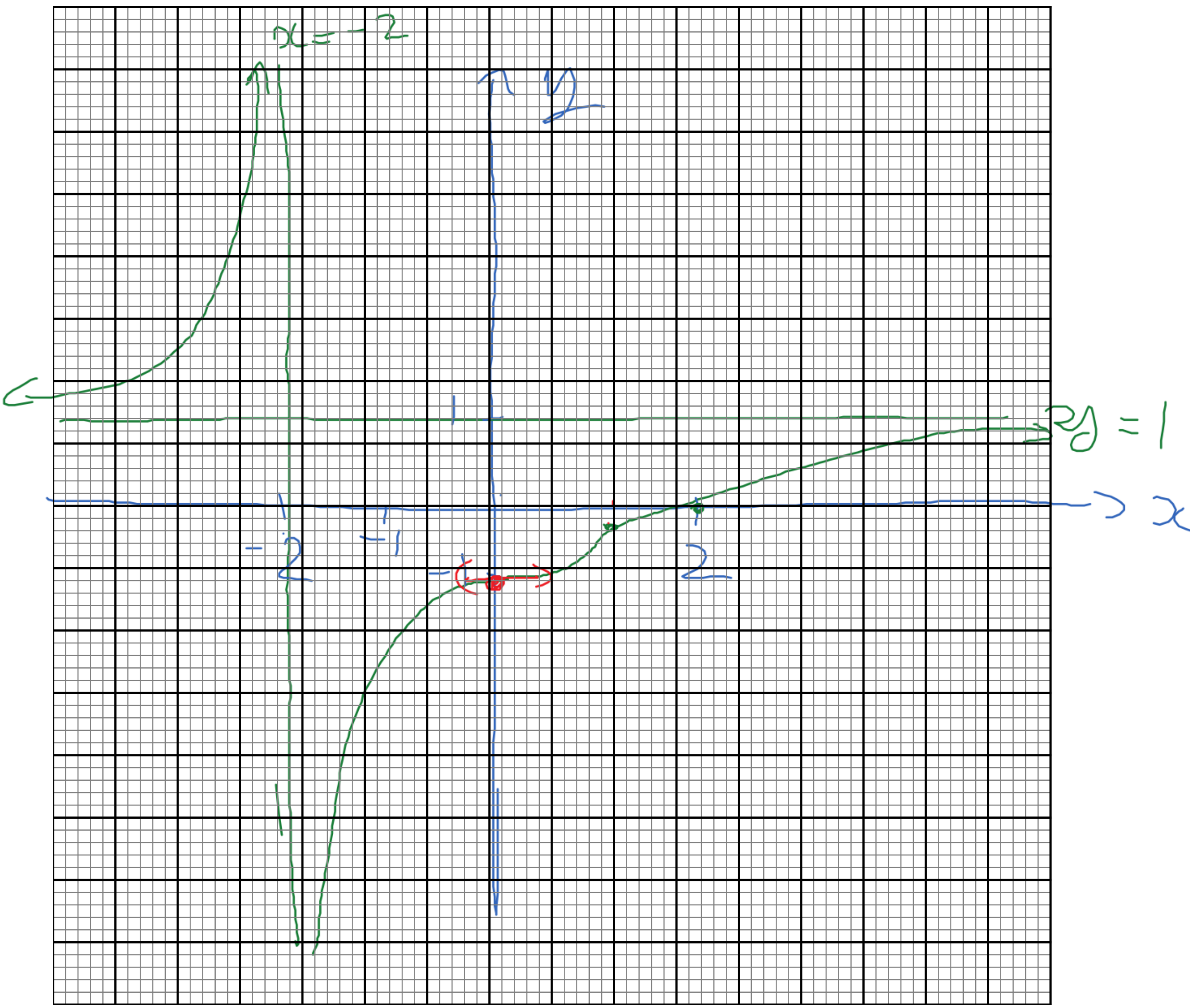
Concave up on $(-\infty,-2)$ and $(0,\sqrt[3]{4})$

Concave down on $(-2,0)$ and $(\sqrt[3]{4},\infty)$

Inflection points: $(0,-1)$ and $(\sqrt[3]{4},-1/3)$ (change in concavity)

$f(x) = \dfrac{x^3 - 8}{x^3 + 8}$ $\quad f'(x) = \dfrac{48x^2}{(x^3 + 8)^2}$ $\quad f''(x) = \dfrac{-192x(x^3 - 4)}{(x^3 + 8)^3}$

Continued



[5 pts] 11. The total resistance R of two parallel resistors R_1 and R_2 is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 is increasing at a rate of $0.21\,\Omega/s$ (ohm per second), while R_2 is decreasing at a rate of $0.12\,\Omega/s$, find the rate of change of the total resistance R when R_1 is $6\,\Omega$ and R_2 is $3\,\Omega$.

$\frac{dR_1}{dt} = 0.21\,\Omega/s \qquad \frac{dR_2}{dt} = -0.12\,\Omega/s$

$$\frac{d}{dt} R^{-1} = \frac{d}{dt} R_1^{-1} + \frac{d}{dt} R_2^{-1}$$

$$-1R^{-2}\frac{dR}{dt} = -R_1^{-2}\frac{dR_1}{dt} - R_2^{-2}\frac{dR_2}{dt}$$

When $R_1 = 6\,\Omega$ and $R_2 = 3\,\Omega$, $\frac{1}{R} = \frac{1}{6} + \frac{1}{3} = \frac{1+2}{6} = \frac{1}{2}$

$\Rightarrow R = 2$

$\Rightarrow -\frac{1}{4}\frac{dR}{dt} = -\frac{1}{36}\cdot 0.21 - \frac{1}{9}(-0.12)$

$\Rightarrow \quad \frac{dR}{dt} = 4\left(\frac{0.7}{12} - \frac{0.4}{3}\right) = 4\left(\frac{0.7-1.6}{12}\right) = \frac{4(-0.9)}{12} = -0.3\,\Omega/s$

R decreases at $0.3\,\Omega/s$

[4 pts] 12. Find the absolute minimum and maximum of $f(x) = x^4 - 8x^2 + 7$ on $[-3, 1]$.

f is continuous on $[-3, 1]$

f is differentiable on $(-3, 1)$ and $f'(x) = 0 \Leftrightarrow 4x^3 - 16x = 0$

$4x(x-2)(x+2)$

$x = 0 \qquad \cancel{x = 2} \notin [-3, 1] \qquad x = -2$

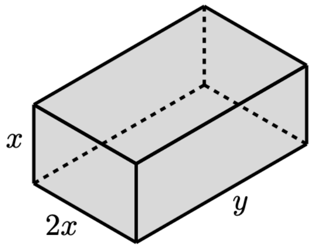
$f(-3) = (-3)^4 - 8(-3)^2 + 7 = 81 - 72 + 7 = 16 \leftarrow$ absolute maximum 16

$f(-2) = (-2)^4 - 8(-2)^2 + 7 = 16 - 32 + 7 = -9 \leftarrow$ absolute minimum -9

$f(0) = 7$

$f(1) = 1 - 8 + 7 = 0$

[5 pts] 13. Canadian postal regulation requires that the sum of the three dimensions (length+width+height) of a rectangular package to be $3\,m$. If the lenght of a package is twice the width (see figure below), find the dimension of the package of maximum volume that can be mailed.



$V = x(2x)y = 2x^2 y$

$x + 2x + y = 3 \Rightarrow 3x + y = 3$

$\Rightarrow y = 3(1-x)$

$x \geq 0$ and $y \geq 0 \Rightarrow 1-x \geq 0 \Rightarrow x \leq 1$

$V(x) = 2x^2(3(1-x)) = 6(x^2 - x^3)$

on $[0,1]$

$V'(x) = 6(2x - 3x^2) = 6x(2-3x)$

V is continuous on $[0,1]$ and differentiable on $(0,1)$

$V'(x) = 0 \Leftrightarrow x = 0$ or $2 - 3x = 0$

$\hookrightarrow x = \frac{2}{3}$

$V(0) = 0$

$V(\frac{2}{3}) = 6\left(\frac{4}{9} - \frac{8}{27}\right) = 6 \cdot \frac{4}{27} = \frac{8}{9} \leftarrow$ Maximum Volume

$V(1) = 0$

$\Rightarrow x = \frac{2}{3}$, $2x = \frac{4}{3}$, $y = 3(1-\frac{2}{3}) = 1$

The dimensions are $\frac{2}{3}$ m by $\frac{4}{3}$ m by 1 m.

[6 pts] 14. Find the following integrals

(a) $\int \frac{x^2+1}{x} dx = \int (x + x^{-1}) dx = \frac{x^2}{2} + \ln|x| + C$

(b) $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + C$

$= \frac{1}{2} \ln(x^2+1) + C$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

[5 pts] 15. The velocity, $v(t)$ of a particle moving in straight line is given by the **separable** differential equation:

$$t\frac{dv}{dt} = (1 + 2t^2)v$$

Find the velocity, $v(t)$, and the position function, $s(t)$, of the particle knowing that $v(1) = 2e$ and $s(0) = 0$.

$$\frac{1}{v}\frac{dv}{dt} = \frac{1+2t^2}{t}$$

$$\int \frac{1}{v}\,dv = \int \left(\frac{1}{t} + 2t\right)dt$$

$$\ln|v| = \ln|t| + t^2 + C$$

$$|v| = e^{\ln|t| + t^2 + C} = e^C t e^{t^2}$$

$$v = K t e^{t^2}$$

$$v(1) = K e = 2e \Rightarrow K = 2$$

$$\boxed{v(t) = 2te^{t^2}}$$

$$s(t) = \int 2te^{t^2}\,dt \qquad u = t^2, \quad du = 2t\,dt$$

$$s(t) = \int e^u\,du = e^u + C$$

$$s(t) = e^{t^2} + C$$

$$s(0) = 1 + C = 0 \Rightarrow C = -1$$

$$\boxed{s(t) = e^{t^2} - 1}$$

Supplementary page