STUDENT'S NAME: ____

STUDENT'S ID NUMBER: ______________________________________

INSTRUCTOR'S NAME: ______________________________________

DAWSON COLLEGE – DEPARTMENT OF MATHEMATICS
LINEAR ALGEBRA (201-NYC-05) COMPUTER SCIENCE **FINAL EXAM** **THURSDAY, DECEMBER 16, 2021 (FROM 14:00 TO 17:00)**
INSTRUCTORS: K. AMEUR, A. JUHASZ, A. JIMENEZ **SECTIONS : 9, 10, 11**

<u>NOTE</u>: This exam has 17 questions for a total of 100 marks. No information sheet is provided.
SHOW ALL THE STEPS! NO MARKS ARE GIVEN FOR FINAL ANSWERS ONLY!

INSTRUCTIONS:
Print your name, ID number and instructor's name in the space provided above.
All questions are to be answered in the space provided for each question.
Give detailed solutions and explain your work clearly.
Reverse sides of pages may be used for rough work and/or to complete a solution.
Only calculator permitted is SHARP EL-531***.
This examination consists of 17 questions on 17 pages. Please ensure that you
have a complete examination before starting.

(5 Marks)
Q1:
a) Find the general solution of the following linear system.

$$\begin{cases} x_1+2x_2+x_3+4x_4=1 \\ 3x_1+6x_2+5x_3+18x_4=1 \\ x_1+2x_2+2x_3+7x_4=0 \end{cases}$$

(2 Marks)

b) Find a particular solution for which $x_1=1$ and $x_3=2$.

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 1 & 4 & 1 \\ 3 & 6 & 5 & 18 & 1 \\ 1 & 2 & 2 & 7 & 0 \end{array}\right) \begin{array}{l} \\ -3R_1 \\ -R_1 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 1 \\ 0 & 0 & 2 & 6 & -2 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{2}R_2 \end{array}$$

$$\begin{array}{ccccc} x_1 & s & x_3 & t & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right)$$

General solution

$$\begin{cases} x_4=t \\ x_3=-2-3t \\ x_2=s \\ x_1=1-4t-x_3-2s \\ \quad =1-4t+2+3t+2s \\ \quad =3-t+2s \end{cases}$$

$x_3=2 \Rightarrow 2=-2-3t \Rightarrow t=-\frac{4}{3}$

$x_1=1 \Rightarrow 1=3+\frac{4}{3}+2s \Rightarrow s=-\frac{5}{3}$

particular solution:

$(x_1, x_2, x_3, x_4)=\left(1, -\frac{5}{3}, 2, -\frac{4}{3}\right)$

(6 Marks)

Q2: Determine the value(s) of k for which the following system has

a) no solutions (2 marks)

b) infinitely many solutions (2 marks)

c) exactly one solution (2 marks)

$$\begin{cases} x & +y & -z & = & 3 \\ 2x & +3y & +z & = & 8 \\ 3x & +6y & +(k^2+2)z & = & k+13 \end{cases}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 2 & 3 & 1 & 8 \\ 3 & 6 & k^2+2 & k+13 \end{array}\right) \begin{array}{l} \\ -2R_1 \\ -3R_1 \end{array} \rightarrow \left(\begin{array}{cccc} 1 & 1 & -1 & 3 \\ 0 & \boxed{1} & 3 & 2 \\ 0 & 3 & k^2+5 & k+4 \end{array}\right) -3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right)$$

a) 0 $\neq 0$

b) 0 0

c) $\neq 0$ any value

a) $k^2-4=0$ and $k-2\neq 0$ $\Rightarrow$ $\boxed{k=-2}$

b) $k^2-4=0$ and $k-2=0$ $\Rightarrow$ $\boxed{k=2}$

c) $k^2-4\neq 2$ $\Rightarrow$ $\boxed{k\in\mathbb{R}/\{-2,2\}}$

(5 Marks)

Q3: Solve for X the following matrix equation.

$$\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}X^{-1}-2I\right)^{-1}=\begin{bmatrix}1 & 4\\1 & 3\end{bmatrix}$$

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}X^{-1}-2I=\begin{pmatrix}1 & 4\\1 & 3\end{pmatrix}^{-1}$$

$$\Rightarrow \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}X^{-1}=\begin{pmatrix}1 & 4\\1 & 3\end{pmatrix}^{-1}+2I$$

$$\Rightarrow X\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{-1}=\left(\begin{pmatrix}1 & 4\\1 & 3\end{pmatrix}^{-1}+2I\right)^{-1}$$

$$\Rightarrow X=\left(\begin{pmatrix}1 & 4\\1 & 3\end{pmatrix}^{-1}+2I\right)^{-1}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$=\left(\begin{pmatrix}-3 & 4\\1 & -1\end{pmatrix}+\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\right)^{-1}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 4\\1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}=-\frac{1}{5}\begin{pmatrix}1 & -4\\-1 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$=-\frac{1}{5}\begin{pmatrix}1 & -2\\-1 & -3\end{pmatrix}=\begin{pmatrix}-\frac{1}{5} & \frac{2}{5}\\\frac{1}{5} & \frac{3}{5}\end{pmatrix}$$

(4 Marks)

Q4: If $A=\begin{bmatrix}1&4&2\\-3&-1&1\\0&0&1\end{bmatrix}$ and $B=\begin{bmatrix}-1&-4&-2\\0&11&7\\0&0&1\end{bmatrix}$, then find two elementary matrices

E_1, E_2 such that

$$E_2E_1A=B$$

$$\underbrace{\begin{pmatrix}1&4&2\\-3&-1&1\\0&0&1\end{pmatrix}}_{A} \xrightarrow{3R_1} \begin{pmatrix}1&4&2\\0&11&7\\0&0&1\end{pmatrix} \xrightarrow{(-1)} \underbrace{\begin{pmatrix}-1&-4&-2\\0&11&7\\0&0&1\end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}}_{E_1} \qquad \underbrace{\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}}_{E_2}$$

(4 Marks)

Q5: If D, E and F are invertible matrices, then simplify the following expression.

$$\left(6F^{-1}E(D^T)^{-1}\right)^T\left(3D^{-2}E^T(F^{-1})^T\right)^{-1}$$

$$6((D^T)^{-1})^T E^T (F^{-1})^T \frac{1}{3}((F^{-1})^T)^{-1}(E^T)^{-1}(D^{-2})^{-1}$$

$$= 2D^{-1}\underbrace{E^T\underbrace{(F^{-1})^T((F^{-1})^T)^{-1}}_{I}(E^T)^{-1}}_{I}D^2$$

$$= 2D^{-1}D^2 = 2D$$

(3 Marks)

Q6: Assuming that $A^3 = 3A^2 - 2A + I$, then show that A is invertible and express A^{-1} in terms of A.

$$A^3 - 3A^2 + 2A = I$$

$$A\underbrace{(A^2 - 3A + 2I)}_{A^{-1}} = I$$

$$A^{-1} = A^2 - 3A + 2I$$

(3 Marks)

Q7: Determine if the following statement is true or false. If it is true prove it and if it is false, then show it with a counter-example.

If A and B are square matrices, then $det(A+B)=det(A)+det(B)$.

False $\quad A=B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$A+B=\begin{pmatrix}2&0\\0&2\end{pmatrix}$

$det(A+B)=4 \neq det(A)+det(B)=2$

Q8: If A and B are 3x3 matrices, det (A) = 2, det(B) =3, and $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then calculate the followings déterminants.

(4 Marks)

a) $det((3A)^{-1}(9B^{-2}A^3)^T) = \dfrac{det(9B^{-2}A^3)}{det(3A)} = \dfrac{9^3 3^{-2} 2^3}{3^3 2}$

$= 3 \cdot 2^2 = 12$

(4 Marks)

b) $det(BA^{-1} - Badj(A)) = det(B)\, det(A^{-1} - adj(A))$

$= det(B)\, det(A^{-1} - A^{-1} det(A)) = \frac{3}{2}(-1)^3 = -\frac{3}{2}$

(4 Marks)

c) $det\begin{bmatrix} a+3d & b+3e & c+3f \\ -2d & -2e & -2f \\ -a+5g & -b+5h & -c+5i \end{bmatrix} = -2 \begin{vmatrix} a & b & c \\ d & e & f \\ -a+5g & -b+5h & -c+5i \end{vmatrix}$

$= -10 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -20$

Q9:

(6 Marks)

a) Use the adjoint method to find the inverse of $A=\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

(2 Marks)

b) Use the result you obtained in part a) to solve the following linear system.

$$\begin{cases} 2x \quad\quad\; +3z = 3 \\ \quad\quad 3y +2z = -2 \\ -2x \quad\quad -4z = 1 \end{cases}$$

a) $|A| = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = -6$

$$adj(A) = \begin{pmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{pmatrix}^T = \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{6} \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}$$

b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \cdot 3 - 9 \cdot 1 \\ -4 \cdot 3 + 4 - 4 \\ 18 + 6 \end{pmatrix} \left(-\frac{1}{6}\right)$

$$= \begin{pmatrix} -45 \\ -12 \\ 24 \end{pmatrix} \left(-\frac{1}{6}\right) = \begin{pmatrix} \frac{15}{2} \\ 2 \\ -4 \end{pmatrix}$$

(3 Marks)

Q10: Solve the following system only for the "x" variable using Cramer's rule.

$$\begin{cases} 2x & +y & -z & = & 0 \\ x & -y & +z & = & 6 \\ x & +2y & +z & = & 3 \end{cases}$$

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{vmatrix} = -\begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix} = -9$$

$$|A_x| = \begin{vmatrix} 0 & 1 & -1 \\ 6 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 6 & 0 & 1 \\ 3 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 6 & 0 \\ 3 & 3 \end{vmatrix} = -18$$

$$x = \frac{|A_x|}{|A|} = 2$$

Q11: Let $\vec{u}=(2,1,4)$, $\vec{v}=(-3,1,0)$ and $\vec{W}=(1,0,1)$, then answer the followings questions.

(3 Marks)

a) The area of the triangle determined by $\vec{u}$ and $\vec{v}$.

$$A_\Delta = \frac{\|\vec{u}\times\vec{v}\|}{2}$$

(3 Marks)

b) The components of $\vec{u}=(2,1,4)$ orthogonal to $\vec{v}+\vec{w}$.

$$\vec{u} - \frac{\vec{u}\cdot(\vec{v}+\vec{w})}{\|v+w\|^2}(\vec{v}+\vec{w})$$

(3 Marks)

c) The volume of the parallelepiped determined by the vectors $\vec{u},\vec{v},\vec{w}$.

$$V = |\vec{w}\cdot(\vec{u}\times\vec{v})|$$

a) $\vec{u}\times\vec{v} = (-4, -12, 5)$

$\|\vec{u}\times\vec{v}\| = \sqrt{4^2+12^2+5^2} = \sqrt{185}$

$$\boxed{A_\Delta = \frac{\sqrt{185}}{2}}$$

b) $\vec{v}+\vec{w} = (-2, 1, 1)$

$\|\vec{v}+\vec{w}\|^2 = 2^2+1^2+1^2 = 6$

$\vec{u}\cdot(\vec{v}+\vec{w}) = 2(-2)+1\cdot 1+4\cdot 1 = 1$

$\vec{u} - \frac{1}{6}(-2, 1, 1) = (2, 1, 4) - \left(\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right)$

$$= \boxed{\left(\frac{5}{3}, \frac{5}{6}, \frac{23}{6}\right)}$$

c) $\vec{u}\times\vec{v} = (-4, -12, 5)$

$\vec{w}\cdot(\vec{u}\times\vec{v}) = 1(-4) + 0(-12) + 1.5 = 1$

$V = 1$

Q12: Given point A(2,-1,3), plane P: 2x-y+4=0 and line L: (x, y, z) =(3+t, -1-2t, 1-t) , then answer the following questions.

(4 Marks)

a) Find the distance from A to L. $\left\| \vec{BA} - \frac{\vec{BA}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \right\| = D(A,L)$

length of components of $\vec{BA}$ orthogonal to $\vec{v}$

(4 Marks) $B(3,-1,1)$ $\vec{v}=(1,-2,-1)$

b) Find the distance from A to P.

(4 Marks)

c) Find the closest point to A on the plane P. // Intersection of $A+(2,-1,0)t$ line with P plane

a) $\vec{BA} = (-1,0,2)$

$\vec{BA}\cdot\vec{v} = -1\cdot 1 + 0(-2) + 2(-1) = -3$

$\|\vec{v}\|^2 = 1^2+2^2+1^2 = 6$

$(-1,0,2) + \frac{1}{2}(1,-2,-1) = \left(-\frac{1}{2},-2,1\right)$

$D(A,L) = \sqrt{\frac{1}{4}+4+1} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$

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b) $D(A,P) = \frac{|2\cdot 2-(-1)+4|}{\sqrt{4+1}} = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$

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c) $A+(2,-1,0)t = (2+2t, -1-t, 3)$

$2(2+2t)-(-1-t)+4=0$ Solve for t

$9+5t=0 \Rightarrow t = -\frac{9}{5}$

So, closest point

$\left(2+2\left(-\frac{9}{5}\right), -1-\left(-\frac{9}{5}\right), 3\right) = \left(-\frac{8}{5}, \frac{4}{5}, 3\right)$

(4 Marks)

Q13: Find the equation of the line of intersection of the planes x-2y+5z=1 and 2x-5y+4z=3.

$$\begin{cases} x - 2y + 5z = 1 \\ 2x - 5y + 4z = 3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -5 & 4 & 3 \end{array}\right)_{-2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & -1 & -6 & 1 \end{array}\right)_{(-1)} \rightarrow \begin{array}{c} \begin{array}{ccc} x & y & t \end{array} \\ \left(\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 1 & 6 & -1 \end{array}\right) \end{array}$$

$$\begin{cases} z = t \\ y = -1 - 6t \\ x = 1 - 5t + 2y \\ \quad = 1 - 5t - 2 - 12t \\ \quad = -1 - 17t \end{cases}$$

$$\boxed{(-1, -17t, -1-6t, t) = (x, y, z)}$$

or

$$P = P(-1, -1, 0) + (-17, -6, 1)t$$

(5 Marks)

Q14: Calculate the distance between the skew lines L1 and L2 where:

L1: (x, y, z) = (2-3t, 4, 2-2t)

L2: (x, y, z) = (1+7s, 2-s, 3+5s)

$$\|proj_{\vec{v}}(\vec{AB})\| = D(L_1, L_2)$$

where $A(2,4,2)$, $B(1,2,3)$

and $\vec{v} = (-3,0,-2) \times (7,-1,5)$

$= (-2, 1, 3)$

$$\vec{AB} = (-1, -2, 1)$$

$$D(L_1, L_2) = \frac{|(-1,-2,1)\cdot(-2,1,3)|}{\sqrt{2^2+1^2+3^2}}$$

$$= \frac{|2-2+3|}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

(5 Marks)

Q15: Write the equation of the plane containing the line (x, y, z) = (1-2t, 3t, 5+t) and is perpendicular to plane x+3y-z=2.

$\bar{n} = (-2, 3, 1) \times (1, 3, -1) = (-6, -1, -9)$

$P_0 = (1, 0, 5)$

$\boxed{-6(x-1) - y - 9(z-5) = 0}$

(5 Marks)

Q16: Maximize $P = 7x_1 + 8x_2 + 10x_3$

subject to
$$\begin{cases} 2x_1 + 3x_2 + 2x_3 \le 10 \\ x_1 + x_2 + 2x_3 \le 80 \end{cases}$$
$$x_1, x_2, x_3 \ge 0$$

$$\left.\begin{array}{ccc} 2 & 3 & 2 \\ 1 & 1 & 2 \\ \hline -7 & -8 & -10 \end{array}\right| \begin{array}{c} 10 \\ 80 \\ \hline 0 \end{array}$$

$$\begin{array}{ccc} & & \uparrow \end{array}$$

$$\left.\begin{array}{ccc} 1 & \frac{3}{2} & 1 \\ -1 & -2 & 0 \\ \hline 3 & 7 & 0 \end{array}\right| \begin{array}{c} 5 \\ 70 \\ \hline 50 = P \end{array}$$

$$\begin{cases} x_1 = 0, \; x_2 = 0, \; x_3 = 5 \\ P = 7 \cdot 0 + 8 \cdot 0 + 10 \cdot 5 = 50 \end{cases}$$

(5 Marks)

Q17: Minimize $C = 9x_1 + 2x_2 + 5x_3$

subject to
$$\begin{cases} x_1 + x_2 \geq 6 \\ 3x_1 + 6x_2 + 2x_3 \geq 2 \\ x_1 - x_2 + x_3 \geq 8 \end{cases}$$
$$x_1, x_2, x_3 \geq 0$$

$$A = \begin{array}{cccc} 1 & 1 & 0 & 6 \\ 3 & 6 & 2 & 2 \\ 1 & -1 & 1 & 8 \\ 9 & 2 & 5 & \cancel{*} \end{array}$$

$$A^T = \begin{pmatrix} 1 & 3 & 1 & 9 \\ 1 & 6 & -1 & 2 \\ 0 & 2 & 1 & 5 \\ 6 & 2 & 8 & * \end{pmatrix}$$

$$\begin{array}{ccc|ccc|c|l} 1 & 3 & 1 & 1 & 0 & 0 & 9 & \\ 1 & 6 & -1 & 0 & 1 & 0 & 2 & +R_1 \\ 0 & 2 & \boxed{1} & 0 & 0 & 1 & 5 & -R_1 \\ \hline -6 & -2 & -8 & 0 & 0 & 0 & 0 & 8R_1 \end{array}$$

$$\begin{array}{ccc|ccc|c} 1 & 3 & 1 & 1 & 0 & 0 & 9 \\ 2 & 9 & 0 & 1 & 1 & 0 & 11 \\ 0 & -1 & 0 & -1 & 0 & 1 & -4 \\ \hline 2 & 22 & 0 & 8 & 0 & 0 & 72 \end{array}$$

$x_1 = 8 \quad x_2 = 0, \; x_3 = 0 \qquad C = 9 \cdot 8 + 2 \cdot 0 + 5 \cdot 0 = 72$